Math 564: Advance Analysis 1
Lecture 4

Prop. For an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ and a premeasure μ on $\mathcal{A}$, then $\mu^*|_{\mathcal{A}} = \mu$.

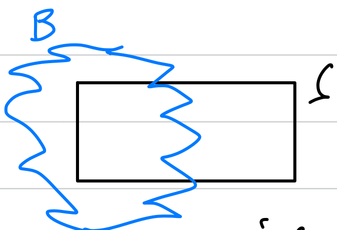
Proof. Let $A \in \mathcal{A}$, we need to show that for any cover $\{A_n\} \subseteq \mathcal{A}$ of A $\mu(A) \leq \sum_n \mu(A_n)$. WLOG, assume $A_n \subseteq A$ by replacing A_n with $A_n \cap A$ and noting that $\mu(A_n \cap A) \leq \mu(A_n)$ by monotonicity.
Now $\bigcup_n A_n = A$. By disjointifying, we may assume $\bigsqcup_n A_n = A$, so ctbl additivity gives $\mu(A) = \sum_n \mu(A_n)$. □

We are now ready to prove

Carathéodory's extension theorem. Every premeasure μ on an algebra $\mathcal{A}$ admits an extension to a measure on $\langle \mathcal{A} \rangle_\sigma$.
If μ is σ-finite, then this extension is unique and still denote it μ.

Proof of existence. It is enough to show that μ^* is finitely additive on $\langle \mathcal{A} \rangle_\sigma$ because then it would be ctbly superadditive, hence ctbly additive because μ^* is already ctbly subadditive.

Proof 1 (Carathéodory). Say that a set $B \subseteq X$ **butchers** a set $S \subseteq X$ if



$$\mu^*(S) \overset{>}{\neq} \mu^*(B \cap S) + \mu^*(B^c \cap S).$$

Let $\mathcal{B}$ be the collection of all non-butchering sets, i.e. sets that don't butcher any other set. Then one shows that (a) $\mathcal{B}$ is a σ-algebra.
(b) $\mathcal{B} \supseteq \mathcal{A}$. (Easy.)

Then it's automatic that μ^* is finitely additive on $\mathcal{B}$. □

First suppose that $\mu(X) < \infty$.

Proof 2 (Tao). We define a pseudo-metric d on $\mathcal{P}(X)$ by
$$d(A,B) := \mu^*(A \triangle B).$$

Pseudo-metric is almost a metric except that $d(A,B) = 0 \not\Rightarrow A = B$.

Detour on $\triangle$. $(\mathcal{P}(X), \triangle)$ is an abelian group with $\emptyset$ being the 0 and
$\forall A \in \mathcal{P}(X)$, $A \triangle A = \emptyset$.
IS
$(2^X, +_2)$, where $+_2$ is the xor / binary addition.

Recall: $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Claim (a). Triangle inequality holds: $d(A,C) \le d(A,B) + d(B,C)$.

Proof. $A \triangle C = A \triangle (B \triangle B) \triangle C = (A \triangle B) \triangle (B \triangle C) \subseteq (A \triangle B) \cup (B \triangle C)$, so
$$d(A,C) = \mu^*(A \triangle C) \underset{\text{mon}}{\le} \mu^*((A \triangle B) \cup (B \triangle C)) \underset{\text{subadd}}{\le} d(A,B) + d(B,C).$$ $\square$

Let $\mathcal{B} := \overline{\mathcal{A}}^d$, the closure of $\mathcal{A}$ in $\mathcal{P}(X)$ with respect to d.

We show that $\mathcal{B}$ is a σ-algebra and μ^* is finitely additive on $\mathcal{B}$.

Claim (b). The function $\begin{array}{c} \mathcal{P}(X) \to [0,\infty) \\ A \mapsto \mu^*(A) \end{array}$ is 1-Lipschitz wrt d, i.e.
$$|\mu^*(A) - \mu^*(B)| \le d(A,B) \quad \text{for any } A, B \in \mathcal{P}(X).$$
In particular, it's continuous: $A_n \to_d A \Rightarrow \mu^*(A_n) \to \mu^*(A)$.

Proof. $\mu^*(A) = d(A, \emptyset)$ and $x \mapsto d(x, x_0)$ is always 1-Lipschitz:
$|d(x,x_0) - d(y,x_0)| \le d(x,y)$ by triangle inequality. $\square$

Claim (c). The map $\mathcal{P}(X) \to \mathcal{P}(X)$, $A \mapsto A^c$ is an isometry, in partic. continuous.

Proof. $A \triangle B = A^c \triangle B^c$, hence $d(A,B) = d(A^c, B^c)$. $\square$

Thus, $\mathcal{B}$ is closed under complements because $A_n \to A \Rightarrow A_n^c \to A^c$ so if $A \in \mathcal{B}$ and $A_n \in \mathcal{A}$ and $A_n \to A$, then $A_n^c \in \mathcal{A}$ and $A_n^c \to A^c$ so $A^c \in \mathcal{B}$.

Claim (d). The map $\mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$, $(A,B) \mapsto A \cup B$ is 1-Lipschitz wrt "$d+d$" metric on $\mathcal{P}(X)^2$, i.e.

$$d(A_1 \cup B_1, A_2 \cup B_2) \leq d_1^{(2)}((A_1,B_1),(A_2,B_2)) := d(A_1,A_2) + d(B_1,B_2).$$

In particular, $\cup$ is continuous, hence so is $\cap$ (because complement is).

Proof.

$$(A_1 \cup B_1) \triangle (A_2 \cup B_2) \subseteq (A_1 \triangle A_2) \cup (B_1 \triangle B_2), \text{ so}$$

$$d(A_1 \cup B_1, A_2 \cup B_2) \underset{\text{mon}}{\leq} \mu^*((A_1 \triangle A_2) \cup (B_1 \triangle B_2)) \underset{\text{subadd}}{\leq} d(A_1,A_2) + d(B_1,B_2).$$

$\square$

Thus $\mathcal{B}$ is closed under finite unions because if $A, B \in \mathcal{B}$ then $\exists \{A_n, B_n\} \subseteq \mathcal{A}$ with $A_n \to_d A$ and $B_n \to_d B$, hence by continuity of $\cup$, $A_n \cup B_n \to_d A \cup B$ and each $A_n \cup B_n \in \mathcal{A}$.

Claim (d'). μ^* is finitely additive on the algebra $\mathcal{B}$.

Proof. Let $A, B \in \mathcal{B}$ be disjoint, and aim to show $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. Let $A_n \to_d A$ and $B_n \to_d B$ with $A_n, B_n \in \mathcal{A}$. By the continuity of $\cup$, $A_n \cup B_n \to_d A \cup B \overset{(b)}{\Longrightarrow} \mu^*(A_n \cup B_n) \to \mu^*(A \cup B)$.
But $\mu^*(A_n \cup B_n) = \mu(A_n \cup B_n) \underset{\varepsilon_n}{\approx} \mu(A_n) + \mu(B_n)$ because $A_n \cap B_n \to A \cap B = \emptyset \Rightarrow \mu(A_n \cap B_n) \to 0$. Hence:

$$\mu^*(A \cup B) \underset{\varepsilon_n}{\approx} \mu^*(A_n \cup B_n) \underset{\varepsilon_n}{\approx} \mu^*(A_n) + \mu^*(B_n) \underset{\varepsilon_n}{\approx} \mu^*(A) + \mu^*(B).$$

$\square$

Claim (e). For pairwise disjoint $\{A_n\} \subseteq \mathcal{A}$, $\bigsqcup_{n<N} A_n \to_d \bigsqcup_{n\in\mathbb{N}} A_n \; (=: A)$. In particular, $\mathcal{B}$ contains all ctbl unions of sets in $\mathcal{A}$, by disjointification.

Proof. $d\left(\bigsqcup_{n<N} A_n, A\right) = \mu^*\left(\bigsqcup_{n\geq N} A_n\right) \overset{\text{ctbl subadd}}{\leq} \sum_{n\geq N} \mu(A_n) \to 0$ because

$\sum_{n\in\mathbb{N}} \mu(A_n)$ converges: $\mu(X) \geq \mu^*\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) \underset{\text{mon}}{\geq} \mu^*\left(\bigsqcup_{n<N} A_n\right) = \sum_{n<N} \mu^*(A_n)$. $\square$

Claim (e'). $\mathcal{B}$ is a σ-algebra.

Proof. We only need to show closure under ctbl unions. Let $B_n \in \mathcal{B}$. Let $A_n \in \mathcal{A}$ be s.t. $A_n \approx^d_{\frac{\varepsilon}{2^{n+1}}} B_n$. Then $d\left(\bigcup_{n\in\mathbb{N}} A_n, \bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_n d(A_n, B_n)$ $= \varepsilon$. But $\bigcup_n A_n \in \mathcal{B}$ so since ε is arbitrary and $\mathcal{B}$ is closed, $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. $\square$

This finishes the proof for finite premeasures. The proof for σ-finite premeasures is just that if $X = \bigsqcup_{n\in\mathbb{N}} X_n$ with $\mu(X_n) < \infty$, $X_n \in \mathcal{A}$, then the existence of the extension μ to $\langle \mathcal{A} \rangle_\sigma$ follows from that of $\langle \mathcal{A}|_{X_n} \rangle_\sigma$ for each n. Tao's proof doesn't work for non-σ-finite premeasures. $\square$

Uniqueness. Let μ be a σ-finite premeasure. In fact it's enough to prove for a finite premeasure μ because if $X = \bigcup_{n\in\mathbb{N}} X_n$ with $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$, then $\mu = \sum_{n\in\mathbb{N}} \mu|_{X_n}$.

Let ν be a measure on $\langle \mathcal{A} \rangle_\sigma$ extending μ. We show that $\nu = \mu^*|_{\langle \mathcal{A} \rangle_\sigma}$.
Firstly, note that $\nu \leq \mu^*$ because if $B \in \langle \mathcal{A} \rangle_\sigma$ and $\bigcup_n A_n \supseteq B$

with $A_n \in \mathcal{A}$, then $\nu(B) \overset{\text{mon.}}{\leq} \nu\left(\bigcup_n A_n\right) \overset{\text{ctbl subadd}}{\leq} \sum_n \nu(A_n) = \sum_n \mu(A_n)$.

$\mu^* = \inf$ of left side, so $\nu(B) \leq \mu^*(B)$.

Next, note that ν is 1-Lipschitz wrt d. Indeed,

$|\nu(A) - \nu(B)| = |\nu(A \setminus B) - \nu(B \setminus A)| \leq \nu(A \setminus B) + \nu(B \setminus A) =$
$= \nu(A \triangle B) \leq \mu^*(A \triangle B) = d(A, B)$.

Thus, ν and μ^* are continuous functions on $\langle \mathcal{A} \rangle_\sigma$ that are equal on the dense set $\mathcal{A}$, hence $\nu = \mu^*$. $\square$